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On Steinerians of Quartic Surfaces.

By John N. Van der Vries.

Connected with each algebraic surface in three-dimensional space are a number of covariant surfaces. Among these the Steinerian and the Hessian play an important rôle. The Hessian may be defined as the locus of points which are double points on first polars, or as the locus of points whose polar quadrics have double points. The Steinerian, the reciprocal surface to the Hessian, may be defined as the locus of points whose first polars have double points, or as the locus of points which are double points on polar quadrics, *i. e.*, which are the vertices of these quadric cones.* The Hessians and the Steinerians of surfaces of the third order have been investigated by Salmon, Cayley and others. The Hessian has also been investigated for surfaces of the fourth and higher orders. In two-dimensional geometry, the Hessian and the Steinerian of plane curves have been treated quite fully by Hesse, Steiner, Salmon, Cayley and others. The corresponding work in one-dimensional geometry has been done by Professor Newson.

It is the object of this paper to investigate the properties of the Steinerians of surfaces of the fourth order. These Steinerians are surfaces of order 32; they are therefore not simple surfaces. The work of determining them is, as will be seen later, very complicated, and results have been obtained only in special cases. These special cases are treated in detail, the remainder of the paper being devoted to considering special features of the Steinerians of quartic surfaces having certain singularities.

A. *The Developable Quartic.*

The polars of any point on an algebraic surface with respect to the surface itself touch the surface at the point itself. The tangents to the two branches of the curves in which the tangent plane to the surface at that point intersect the surface or any of its polars down to the polar quadric are respectively coincident. The polar quadric of any parabolic point on the surface with respect to the surface itself is a cone. The point itself is not the vertex of this cone, but the tangent plane to the surface at this point touches the cone along a generating line. Every point on a developable surface is a parabolic point. The polar quadric of every point of this surface will be the tangent cone at such a point; and, as two consecutive generators meet in a point on the cuspidal edge, these tangent cones will have as their vertices the points of the cuspidal edge. It has been shown* that the Hessian of a developable quartic is equal to the square of the quartic itself; *i. e.*, every point of the surface is a point of the Hessian. The points of the Steinerian, therefore, which correspond to the points of the Hessian lying on the developable quartic exclusive of the cuspidal edge, are points of the cuspidal edge. The cuspidal edge of the developable quartic is a unodal line. A unode is a double point whose tangent cone breaks up into two coincident planes, every point of these planes thus being a double point. The double point of the polar quadric which corresponds to a point of the Hessian on the cuspidal edge is therefore any point of the unodal plane at that point. The points of the Steinerian corresponding to points on the cuspidal edge of the Hessian are therefore the points of the osculating plane. As this plane varies from point to point along the cuspidal edge and during its revolution takes in every point of space, it is evident that the Steinerian of the developable quartic fills all space. *There is thus no definite Steinerian.*

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*Salmon: "Geometry of Three Dimensions," 4th edition, pp. 492, 493.

This result can also be shown analytically. The developable quartic is the envelope of tangent lines to a twisted cubic, the coordinates of any point of which are given by

$$x : y : z : s = t^3 : t^2 : t : 1;$$

that is,

$$\left\| \begin{matrix} x & y & z & s \\ t^3 & t^2 & t & 1 \end{matrix} \right\| = 0.$$

The equation of the quartic is obtained by eliminating t from

$$\left\| \begin{matrix} x & y & z & s \\ t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \end{matrix} \right\| = 0.$$

Eliminating t from

$$y - 2zt + st^2 = 0, \quad \text{and} \quad x - 2yt + zt^2 = 0,$$

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*Salmon: "Geometry of Three Dimensions," p. 489.

we obtain as the equation of the quartic

$$\begin{vmatrix} x & -2y & z & 0 \\ 0 & x & -2y & z \\ y & -2z & s & 0 \\ 0 & y & -2z & s \end{vmatrix} = 0;$$

i. e.,

$$U \equiv (xs - yz)^2 - 4\,(ys - z^2)\,(xz - y^2) = 0;$$

$$xs - yz = ys - z^2 = xz - y^2 = 0$$

being the equations of the twisted cubic. Let $(\xi, \eta, \zeta, \sigma)$ be the coordinates of a point on the Steinerian, and (x, y, z, s) those of a point on the Hessian. The equation of the first polar of $(\xi, \eta, \zeta, \sigma)$ with respect to the surface $U = 0$ is

$$\xi L + \eta M + \zeta N + \sigma P = 0, \tag{I}$$

where L, M, N, P are the first derivatives of U with respect to x, y, z, and s. If the first polar of $(\xi, \eta, \zeta, \sigma)$ with respect to $U = 0$ has a double point, this double point must satisfy the derivatives of (I) with respect to x, y, z and s; viz.:

$$\begin{aligned}
s(\xi s - \eta z - \zeta y + \sigma x) - 2z(\eta s - 2\zeta z + \sigma y) + \sigma(xs - yz) - 2\zeta(ys - z^2) &= 0, \\
-z(\xi s - \eta z - \zeta y + \sigma x) - 2s(\xi z - 2\eta y + \zeta x) + 4y(\eta s - 2\zeta z + \sigma y) \qquad & \\
- \zeta(xs - yz) + 4\eta(ys - z^2) - 2\sigma(xz - y^2) &= 0, \\
-y(\xi s - \eta z - \zeta y + \sigma x) + 4z(\xi z - 2\eta y + \zeta x) - 2x(\eta s - 2\zeta z + \sigma y) \qquad & \\
- \eta(xs - yz) - 2\xi(ys - z^2) - 4\zeta(xz - y^2) &= 0, \\
x(\xi s - \eta z - \zeta y + \sigma x) - 2y(\xi z - 2\eta y + \zeta x) + \xi(xs - yz) - 2\eta(xz - y^2) &= 0.
\end{aligned}$$

The equation of the Hessian is obtained by eliminating ξ, η, ζ, and σ from these equations, and the equation of the Steinerian by eliminating x, y, z, and s. For points of the Hessian on the cuspidal edge

$$xs - yz = ys - z^2 = xz - y^2 = 0,$$

and the equations for the determination of the corresponding points of the Steinerian reduce to

$$\left.\begin{aligned}
s(\xi s - \eta z - \zeta y + \sigma x) - 2z(\eta s - 2\zeta z + \sigma y) \qquad\qquad\qquad\qquad &= 0, \\
-z(\xi s - \eta z - \zeta y + \sigma x) + 4y(\eta s - 2\zeta z + \sigma y) - 2s(\xi z - 2\eta y + \zeta x) &= 0, \\
-y(\xi s - \eta z - \zeta y + \sigma x) - 2x(\eta s - 2\zeta z + \sigma y) + 4z(\xi z - 2\eta y + \zeta x) &= 0, \\
x(\xi s - \eta z - \zeta y + \sigma x) \qquad\qquad\qquad\qquad - 2y(\xi z - 2\eta y + \zeta x) &= 0.
\end{aligned}\right\} \tag{II}$$

Considering these equations as linear equations in the linear functions

$$\xi s - \eta z - \zeta y + \sigma x, \quad \eta s - 2\zeta z + \sigma y, \quad \text{and} \quad \xi s - 2\zeta z + \sigma y,$$

it is evident that

$$\left\| \begin{array}{cccc} s & -z & -y & x \\ 0 & -2s & 4z & -2y \\ 2z & 4y & -2x & 0 \end{array} \right\| = 0;$$

that is,

$$\begin{array}{llll} x(xs-yz)-2y(xz-y^2) & & & =0, \\ & z(xz-y^2) & -x(ys-z^2) & =0, \\ & -s(xz-y^2) & +y(ys-z^2) & =0, \\ s(xs-yz) & & -2z(ys-z^2) & =0. \end{array}$$

These equations are not independent but are equivalent to two independent equations, viz.:

$$xs-yz=xz-y^2=ys-z^2=0,$$

i. e.,

$$\frac{x}{y}=\frac{y}{z}=\frac{z}{s}.$$

Thus only two of equations (II) are independent. Considering the first and the fourth, substituting t^3, t^2, t and 1 for x, y, z and s, we have

$$\begin{aligned} (\xi-\eta t-\zeta t^2+\sigma t^3)-2t(\eta-2\zeta t+\sigma t^2) &= 0, \\ t^3(\xi-\eta t-\zeta t^2+\sigma t^3)-2t^3(\xi t-2\eta t^2+\zeta t^3) &= 0. \end{aligned}$$

These are, however, identical and equal to

$$\xi-3\eta t-3\zeta t^2-\sigma t^3=0,$$

the equation of the osculating plane of the twisted cubic, Q. E. D.

B. *The Surface* $ax^2s^2+by^2z^2+xyzs=0$.

This surface has four unodes, (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0) and (1, 0, 0, 0), and four double lines, viz.: $x=y=0$, $x=z=0$, $y=s=0$ and $z=s=0$. The polar of the point $(\xi, \eta, \zeta, \sigma)$ with respect to the surface is

$$\xi(2axs^2+yzs)+\eta(2byz^2+xzs)+\zeta(2by^2z=xys)+\sigma(2ax^2s+xyz)=0.$$

If this polar has a double point, this point must satisfy the equations

$$\begin{array}{lllll} \xi(2as^2) & +\eta(zs) & +\zeta(ys) & +\sigma(4axs+yz) & =0, \\ \xi(zs) & +\eta(2bz^2) & +\zeta(4byz+xs) & +\sigma(xz) & =0, \\ \xi(ys) & +\eta(4byz+xs) & +\zeta(2by^2) & +\sigma(xy) & =0, \\ \xi(4axs+yz) & +\eta(xz) & +\xi(xy) & +\sigma(2ax^2) & =0. \end{array} \qquad \text{(III)}$$

The equation of the Hessian reduces to

$$[2byz+xs]\,[2axs+yz]\,[2ax^2s^2-xyzs(1-12ab)+2by^2z^2]=0.$$

It is thus composed of two quadric surfaces, having the double lines of the quartic as simple lines, and a quartic surface having the same singularities as the original quartic.

The equation of the Steinerian is obtained by eliminating x, y, z and s from equations (III) above; this is accomplished by multiplying the four equations successively by x, y, z and s, and then eliminating the variables from the sixteen equations thus obtained and the four derivatives with respect to x, y, z and s of the equation of the Jacobian. The equation of the Jacobian is

$$J \equiv \begin{vmatrix} 4a\sigma s, & \zeta s+\sigma z, & \eta s+\sigma y, & 4a\xi s+\eta z+\zeta y+4a\sigma x \\ \zeta s+\sigma z, & 4b\zeta z, & \zeta s+4b\eta z+4b\zeta y+\sigma x, & \xi z+\zeta x \\ \eta s+\sigma y, & \xi s+4b\eta z+4b\zeta y+\sigma x, & 4b\eta y, & \xi y+\eta x \\ 4a\xi s+\eta z+\zeta y+4a\sigma x, & \xi z+\zeta x, & \xi y+\eta x, & 4a\xi x \end{vmatrix} = 0;$$

that is,

$$\begin{aligned} J \equiv a_1x^4 &+ b_1y^4 + c_1z^4 + d_1s^4 + a_2x^3y + b_2x^3z + c_2ys^3 + d_2xy^3 + e_2zs^3 + f_2y^3s + g_2xz^3 \\ &+ h_2z^3s + i_2yz^3 + j_2xs^3 + k_2x^3s + l_2y^3z + a_3x^2y^2 + b_3x^2z^2 + c_3y^2s^2 + d_3z^2s^2 \\ &+ e_3x^2s^2 + f_3y^2z^2 + a_4x^2yz + b_4xy^2s + c_4xz^2s + d_4yzs^2 + e_4x^2ys + f_4xy^2z \\ &+ g_4yz^2s + h_4xzs^2 + i_4x^2zs + j_4y^2zs + k_4xyz^2 + l_4xys^2 + m_4xyzs = 0, \end{aligned}$$

where

$a_1 = 16a^2\sigma^4,$
$b_1 = 16b^2\zeta^4,$
$c_1 = 16b^2\eta^4,$
$d_1 = 16a^2\xi^4,$
$a_2 = 128a^2b\zeta\sigma^3,$
$b_2 = 128a^2b\eta\sigma^3,$
$c_2 = 128a^2b\xi^3\zeta,$
$d_2 = 128ab^2\zeta^3\sigma,$
$e_2 = 128a^2b\xi^3\eta,$
$f_2 = 128ab^2\xi\zeta^3,$
$g_2 = 128ab^2\eta^3\sigma,$
$h_2 = 128ab^2\xi\eta^3,$
$i_2 = 48b^2\eta^3\zeta - 8b\xi\eta^2\sigma,$
$j_2 = 48a^2\xi^3\sigma - 8a\xi^2\eta\zeta,$
$k_2 = 48a^2\xi\sigma^3 - 8a\eta\zeta\sigma^2,$
$l_2 = 48b^2\eta\zeta^3 - 8b\xi\zeta^2\sigma,$
$a_3 = 32ab\zeta^2\sigma^2 + 256a^2b^2\zeta^2\sigma^2,$
$b_3 = 32ab\eta^2\sigma^2 + 256a^2b^2\eta^2\sigma^2,$
$c_3 = 32ab\xi^2\zeta^2 + 256a^2b^2\xi^2\zeta^2,$

$$
\begin{aligned}
d_3 &= 32ab\xi^2\eta^2 + 256a^2b^2\xi^2\eta^2,\\
e_3 &= 64a^2\xi^2\sigma^2 - 16a\xi\eta\zeta\sigma,\\
f_3 &= 64b^2\eta^2\zeta^2 - 16b\xi\eta\zeta\sigma,\\
a_4 &= -4\eta\zeta\sigma^2 + 128ab\eta\zeta\sigma^2 + 256a^2b^2\eta\zeta\sigma^2 - 8a\xi\sigma^3,\\
b_4 &= -4\xi\zeta^2\sigma + 128ab\xi\zeta^2\sigma + 256a^2b^2\xi\zeta^2\sigma - 8b\eta\zeta^3,\\
c_4 &= -4\xi\eta^2\sigma + 128ab\xi\eta^2\sigma + 256a^2b^2\xi\eta^2\sigma - 8b\eta^3\zeta,\\
d_4 &= -4\xi^2\eta\zeta + 128ab\xi^2\eta\zeta + 256a^2b^2\xi^2\eta\zeta - 8a\xi^3\sigma,\\
e_4 &= -4\eta\zeta^2\sigma + 16a\xi\zeta\sigma^2 + 256a^2b\xi\zeta\sigma^2 - 16ab\eta\zeta^2\sigma,\\
f_4 &= -4\xi\zeta\sigma^2 + 16b\eta\zeta^2\sigma + 256ab^2\eta\zeta^2\sigma - 16ab\xi\zeta\sigma^2,\\
g_4 &= -4\xi^2\eta\sigma + 16b\xi\eta^2\zeta + 256ab^2\xi\eta^2\zeta - 16ab\xi^2\eta\sigma,\\
h_4 &= -4\xi\eta^2\zeta + 16a\xi^2\eta\sigma + 256a^2b\xi^2\eta\sigma - 16ab\xi\eta^2\zeta,\\
i_4 &= -4\eta^2\zeta\sigma + 16a\xi\eta\sigma^2 + 256a^2b\xi\eta\sigma^2 - 16ab\eta^2\zeta\sigma,\\
j_4 &= -4\xi^2\zeta\sigma + 16b\xi\eta\zeta^2 + 256ab^2\xi\eta\zeta^2 - 16ab\xi^2\zeta\sigma,\\
k_4 &= -4\xi\eta\sigma^2 + 16b\eta^2\zeta\sigma + 256ab^2\eta^2\zeta\sigma - 16ab\xi\eta\sigma^2,\\
l_4 &= -4\xi\eta\zeta^2 + 16a\xi^2\zeta\sigma + 256a^2b\xi^2\zeta\sigma - 16ab\xi\eta\zeta^2,\\
m_4 &= +256a^2b^2\xi\eta\zeta\sigma + 256ab\xi\eta\zeta\sigma - 16a\xi^2\sigma^2 - 16b\eta^2\zeta^2.
\end{aligned}
$$

The Steinerian is

$$
\left|\begin{array}{cccccccccccccccccccc}
4a_1, & d_2, & g_2, & j_2, & 3a_2, & 3b_2, & 3k_2, & 2a_3, & f_4, & b_4, & 2b_3, & k_4, & c_4, & 2e_3, & l_4, & h_4, & 2a_4, & 2e_4, & 2i_4, & m_4\\
a_2, & 4b_1, & i_2, & c_2, & 2a_3, & a_4, & e_4, & 3d_2, & 3l_2, & 3f_2, & k_4, & 2f_3, & g_4, & l_4, & 2c_3, & d_4, & 2f_4, & 2b_4, & m_4, & 2j_4\\
b_2, & l_2, & 4c_1, & e_2, & a_4, & 2b_3, & i_4, & f_4, & 2f_3, & j_4, & 3g_2, & 3i_2, & 3h_2, & h_4, & d_4, & 2d_3, & 2k_4, & m_4, & 2c_4, & 2g^4\\
k_2, & f_2, & h_2, & 4d_1, & e_4, & i_4, & 2e_3, & b_4, & j_4, & 2c_3, & c_4, & g_4, & 2d_3, & 3j_2, & 3c_2, & 3e_2, & m_4, & 2l_4, & 2h_4, & 2d_4\\
0, & 0, & 0, & 0, & 0, & 0, & 4a\sigma, & 0, & 0, & 0, & 0, & 0, & 0, & 2a\xi, & 0, & 0, & \sigma, & \zeta, & \eta, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & 0, & 0, & 0, & 2a\xi, & 0, & 0, & 4a\sigma, & 0, & \eta\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & \eta, & 0, & 0, & 2a\xi, & 0, & 0, & 4a\sigma, & \zeta\\
0, & 0, & 0, & 2a\xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4a\sigma, & \zeta, & \eta, & 0, & 0, & 0, & \sigma\\
0, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & 0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & 0, & 4b\zeta, & 0, & \xi, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4b\zeta, & 0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & \xi\\
0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & 4b\zeta, & \xi, & 0, & 0, & 0, & 0, & 0; & \zeta, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\eta, & \zeta, & 0, & \xi, & 0, & 0, & \sigma, & 4b\zeta\\
0, & 0, & 0, & 0, & \sigma, & 0, & \eta, & 2b\zeta, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4b\eta, & \xi, & 0, & 0\\
0, & 2b\zeta, & 0, & 0, & 0, & 0, & 0, & \sigma, & 4b\eta, & \xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \eta, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\zeta, & 0, & 0, & 4b\eta, & 0, & 0, & 0, & 0, & \sigma, & 0, & \eta, & \xi\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\zeta, & 0, & 0, & 0, & \eta, & \xi, & 0, & 0, & \sigma, & 0, & 4b\eta\\
2a\sigma, & 0, & 0, & 0, & \zeta, & \eta, & 4a\xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \xi, & 0, & 0, & 0\\
0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & \zeta, & \xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \eta, & 4a\xi, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & 0, & 0, & \eta, & \xi, & 0, & 0, & 0, & 0, & \zeta, & 0, & 4a\xi, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & 0, & 0, & 0, & 0, & 4a\xi, & 0, & 0, & 0, & \zeta, & \eta, & \xi
\end{array}\right| = 0.
$$

By continued reduction this reduces to

$$2304a^4b^4(1-4ab)^4\xi^4\eta^4\zeta^4\sigma^4\begin{vmatrix} A, & F, & H, & E \\ G, & B, & E, & J \\ I, & E, & C, & L \\ E, & K, & M, & D \end{vmatrix}=0,$$

where

$$\begin{aligned}
A &\equiv -32\eta\zeta\sigma^2+512ab\eta\zeta\sigma^2-1536a^2b^2\eta\zeta\sigma^2,\\
B &\equiv -32\xi\zeta^2\sigma+512ab\xi\zeta^2\sigma-1536a^2b^2\xi\zeta^2\sigma,\\
C &\equiv -32\xi\eta^2\sigma+512ab\xi\eta^2\sigma-1536a^2b^2\xi\eta^2\sigma,\\
D &\equiv -32\xi^2\eta\zeta+512ab\xi^2\eta\zeta-1536a^2b^2\xi^2\eta\zeta,\\
E &\equiv \quad 16\xi\eta\zeta\sigma+128ab\xi\eta\zeta\sigma-768a^2b^2\xi\eta\zeta\sigma,\\
F &\equiv -16\eta\zeta^2\sigma+64ab\eta\zeta^2\sigma+96a\xi\zeta\sigma^2-384a^2b\xi\zeta\sigma^2,\\
G &\equiv -16\xi\zeta\sigma^2-64ab\xi\zeta\sigma^2+96b\eta\zeta^2\sigma-384ab^2\eta\zeta^2\sigma,\\
H &\equiv -16\eta^2\zeta\sigma-64ab\eta^2\zeta\sigma+96a\xi\eta\sigma^2-384a^2b\xi\eta\sigma^2,\\
I &\equiv -16\xi\eta\sigma^2-64ab\xi\eta\sigma^2+96b\eta^2\zeta\sigma-384ab^2\eta^2\zeta\sigma,\\
J &\equiv -16\xi^2\zeta\sigma-64ab\xi^2\zeta\sigma+96b\xi\eta\zeta^2-384ab^2\xi\eta\zeta^2,\\
K &\equiv -16\xi\eta\zeta^2-64ab\xi\eta\zeta^2+96a\xi^2\zeta\sigma-384a^2b\xi^2\zeta\sigma,\\
L &\equiv -16\xi^2\eta\sigma-64ab\xi^2\eta\sigma+96b\xi\eta^2\zeta-384ab^2\xi\eta^2\zeta,\\
M &\equiv -16\xi\eta^2\zeta-64ab\xi\eta^2\zeta+96a\xi^2\eta\sigma-384a^2b\xi^2\eta\sigma.
\end{aligned}$$

This reduces to

$$\begin{aligned}
4076873488a^4b^4(1-4ab)^{10}\xi^7\eta^7\zeta^7\sigma^7\{8(a\xi^2\sigma^2+b\eta^2\zeta^2+\xi\eta\zeta\sigma)\\
-9(1-4ab)(1+12ab)\xi\eta\zeta\sigma\}=0;
\end{aligned}$$

that is, *each of the four unodal planes seven times and a quartic having all the singularities of the original quartic.*

C. *The Surface* $x^2y^2=(ax+by)^2(xz+ys)$.

This is a quartic with a triple line in which two of the sheets which meet in the triple line unite in a single cuspidal sheet. The four equations which must be satisfied by points in order that they may be double points on the first polars of $(\xi, \eta, \zeta, \sigma)$ are

$$\begin{aligned}
A\xi+C\eta+D\zeta+F\sigma&=0,\\
C\xi+B\eta+E\zeta+G\sigma&=0,\\
D\xi+E\eta&=0,\\
F\xi+G\eta&=0,
\end{aligned}$$

where

$$\begin{aligned}
A &\equiv 2y^2 - 2a^2(xz + ys) - 4az(ax + by),\\
B &\equiv 2x^2 - 2b^2(xz + ys) - 4bs(ax + by),\\
C &\equiv 4xy - 2(bz + as)(ax + by) - 2ab(xz + ys),\\
D &\equiv -(ax + by)^2 - 2ax(ax + by),\\
E &\equiv -2bx(ax + by),\\
F &\equiv -2ay(ax + by),\\
G &\equiv -(ax + by)^2 - 2by(ax + by).
\end{aligned}$$

Eliminating ξ, η, ζ and σ, the equation of the Hessian is found to be

$$\begin{vmatrix} A, & C, & D, & F \\ C, & D, & E, & G \\ D, & E, & 0, & 0 \\ F, & G, & 0, & 0 \end{vmatrix} = 0,$$

which reduce to

$$(ax + by)^8 = 0.$$

Thus for every point of the Hessian $ax + by = 0$, and the corresponding points of the Steinerian satisfy the equations

$$\xi\,[2y^2 - 2a^2(xz + ys)] + \eta\,[4xy - 2ab(xz + ys)] = 0$$

and

$$\xi\,[4xy - 2ab(xz + ys)] + \eta\,[2x^2 - 2b^2(xz + ys)] = 0.$$

These are the equations of the line $\xi = \eta = 0$; *i. e.*, the triple line $x = y = 0$. But, for any point of the Hessian on the triple line $x = y = 0$, the equations for determining the corresponding points of the Steinerian vanish identically; that is, *the Steinerian is composed of all the points in space.*

It can readily be seen by similar reasoning that *the Steinerian of every quartic having a triple line is satisfied by all points in space.*

D. *The Quartic* $u_4 + zu_3 + sv_3 + z^2t_2 + zsu_2 + s^2v_2 = 0$.

In this equation u_4, u_3, etc., are functions of x and y of degrees equal to their respective subscripts, the line xy thus being a nodal line on the quartic. The surface, the Hessian and the Steinerian intersect in this line xy. If the last three terms form a perfect square $(xz + ys)^2$, the line is cuspidal; for every point of the Hessian on this line, the corresponding points of the Steinerian satisfy the equation $z\xi + s\eta = 0$. As the point moves along this scrolar line, the plane $z\xi + s\eta$ revolves about the line and *the Steinerian is thus satisfied by all points in space.*

In case the last three terms reduce to the forms x^2zs or x^2z^2, the cuspidal tangent plane remains the same for all points on it, and the points of the Steinerian corresponding to points on this unodal line are points of this unodal plane. But in these two cases there are either one or two triple points on the line, and *in every such case the points of the Steinerian are all the points in space.*

E. *The Quartic Scroll* $(xz-y^2)^2 - mys(xz-y^2) + s^2(axy+by^2) = 0$.

This scroll has the line $x = y = 0$ as a nodal line and $xz - y^2 = s = 0$ as a nodal conic. The Hessian reduces to

$$\begin{aligned}4(xz-y^2)^3\{m^2(xz-3y^2)+8(axy+by^2)\} &+ 4ms(xz-y^2)^2\{4ax(xz+y^2)\\ +4y^2(axy+by^2)-3m^2y^3+8bxyz\} &+ s^2(xz-y^2)\{16x(ay+3bz)(axy+by^2)\\ -m^2y(4ax-m^2y)(xz+y^2)+6bm^2y^2(2xs-y^2) &- 2(m^4+8b^2)y^4+48a^2x^3z\}\\ +2mys^3\{ax(4ax-m^2y)(xz+y^2) &- 2by[6ax^2z+8bxyz+2axy^2\\ -(m^2+4b)y^3]\} - ys^4\{8ax^3 &- a^2(m^2-4b)x^2y+4b^2m^2y^3\} = 0,\end{aligned}$$

and has the nodal conic as a triple conic and the double line as a quadruple line. For a point of the Hessian on the line $x = y = 0$, the corresponding points of the Steinerian satisfy

$$\xi(2z^2) + \eta(mzs+as^2) = 0$$

and

$$\xi(mzs+as^2) + \eta(2bs^2) = 0,$$

which is the line $\xi = \eta = 0$ (that is, the nodal line), *the original surface, the Hessian and the Steinerian thus intersecting in this line.*

Two points of the Hessian are pinch-points on the quartic; viz.,

$$x:y:z:s = 0:0:-a:m-2\sqrt{b}$$

and

$$x:y:z:s = 0:0:-a:m+2\sqrt{b}.$$

Substituting the coordinates of the first in the equations for determining the Steinerian, the points of the Steinerian corresponding to this point are found to satisfy the equations

$$2a^2\xi + (-2am\sqrt{b}+4ab)\eta = 0$$

and

$$(-2am\sqrt{b}+4ab)\xi + 2b(m-2\sqrt{b})\eta = 0,$$

which reduce to the same equation; viz.,

$$a\xi - \sqrt{b}(m-2\sqrt{b})\eta = 0,$$

the pinch-plane at the point. Similarly, for the other pinch-point.

For points of the Hessian on the nodal conic $xz - y^2 = s = 0$, the equations for determining the points of the Hessian and the Steinerian reduce to

$$2(z\xi - 2y\eta + x\zeta) + my\sigma = 0,$$
$$m(z\xi - 2y\eta + x\zeta) + 2(ax + by)\sigma = 0,$$

which are satisfied by

$$z\xi - 2y\eta + x\zeta = \sigma = 0;$$

that is, the tangent line to the conic at any point of it. As the point passes around the conic, this line sweeps over the plane. *The plane of the nodal conic is thus part of the Steinerian.*

The points $x : y : z : s = 0 : 0 : 1 : 0$

and $x : y : z : s = (m^2 - 4b)^2 : 4a(m^2 - 4b) : 16a^2 : 0$

are pinch-points on this conic. By similar reasoning to the above it is seen that

$$\xi = 0 \quad \text{and} \quad 16a^2\xi - 8a(m^2 - 4b)\eta + (m^2 - 4b)^2\zeta + 2am(m^2 - 4b)\sigma = 0,$$

the pinch-planes at the two points, are also parts of the Steinerian.

Similarly, in the case of the quartic with a nodal conic having for its equation

$$\phi^2 - 4P^2\psi = 0,$$

where

$$\phi \equiv ax^2 + by^2 + cz^2 + ds^2,$$
$$\psi \equiv a_1x^2 + b_1y^2 + c_1z^2 + d_1s^2,$$
$$P \equiv z,$$

for points of the Hessian on the nodal conic $ax^2 + by^2 + da^2 = z = 0$, the corresponding points of the Steinerian satisfy

$$ax\xi + by\eta + ds\sigma = 0, \quad \text{and} \quad \zeta = 0;$$

that is, *lie on the plane of the nodal conic* $z = 0$.

F.

In the case of the ruled quartic

$$x^2(az^2 + 2hzs + bs^2) + 2xy(a'z^2 + 2h'zs + b's^2) + y^2(a''z^2 + 2h''zs + b''s^2) = 0,$$

having the two non-intersecting double lines xy and zs, for points of the Hessian on either nodal line the corresponding points of the Steinerian are on the same double line; that is, *the surface, the Steinerian and the Hessian intersect in the double line.*

The same is true for the quartic

$$x^2y^2 + 2mxyzs + s^2u_2 = 0$$

(where u_2 is a quadratic function of x, y, z and s) having xs and ys as intersecting double lines; and it is also true in the case of every double line on the quartic.

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